

## SPECTRA OF CORONAE

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ABSTRACT. We introduce a new invariant, the *coronal* of a graph, and use it to compute the spectrum of the corona  $G \circ H$  of two graphs  $G$  and  $H$ . In particular, we show that this spectrum is completely determined by the spectra of  $G$  and  $H$  and the coronal of  $H$ . Previous work has computed the spectrum of a corona only in the case that  $H$  is regular. We then explicitly compute the coronals for several families of graphs, including regular graphs, complete  $n$ -partite graphs, and paths. Finally, we use the corona construction to generate many infinite families of pairs of cospectral graphs.

## 1. INTRODUCTION

Let  $G$  and  $H$  be (finite, simple, non-empty) graphs. The *corona*  $G \circ H$  of  $G$  and  $H$  is constructed as follows: Choose a labeling of the vertices of  $G$  with labels  $1, 2, \dots, m$ . Take one copy of  $G$  and  $m$  disjoint copies of  $H$ , labeled  $H_1, \dots, H_m$ , and connect each vertex of  $H_i$  to vertex  $i$  of  $G$ . This construction was introduced by Frucht and Harary [3] with the (achieved) goal of constructing a graph whose automorphism group is the wreath product of the two component automorphism groups. Since then, a variety of papers have appeared investigating a wide range of graph-theoretic properties of coronas, such as the bandwidth [2], the minimum sum [12], applications to Ramsey theory [7], etc. Further, the spectral properties of coronas are significant in the study of invertible graphs. Briefly, a graph  $G$  is invertible if the inverse of the graph's adjacency matrix is diagonally similar to the adjacency matrix of another graph  $G^+$ . Motivated by applications to quantum chemistry, Godsil [4] studies invertible bipartite graphs with a unique perfect matching. In response to his question asking for a characterization of such graphs with the additional property that  $G \cong G^+$ , Simian and Cao [10] determine the answer to be exactly the coronas of bipartite graphs with the single-vertex graph  $K_1$ .

The study of spectral properties of coronas was continued by Barik, et. al. in [1], who found the spectrum of the corona  $G \circ H$  in the special case that  $H$  is regular. In Section 2, we drop the regularity condition on  $H$  and compute the spectrum of the corona of any pair of graphs using a new graph invariant called the coronal. In Section 3, we compute the coronal for several families of graphs, including regular graphs examined in [1], complete  $n$ -partite graphs, and path graphs. Finally, in Section 4, we see that properties of the spectrum of coronas lends itself to finding many large families of cospectral graph pairs.

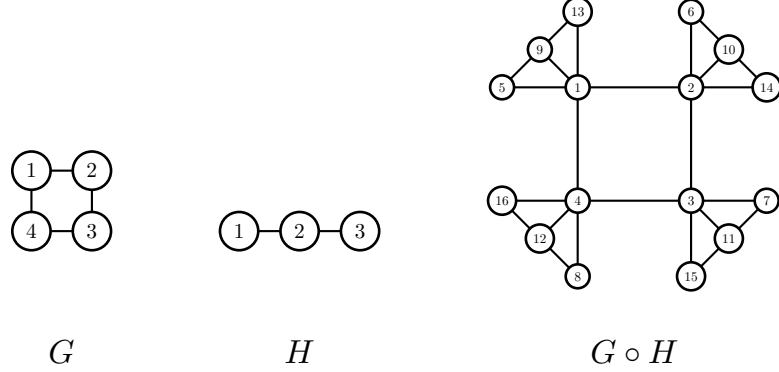
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**Notation.** The symbols  $\mathbf{0}_n$  and  $\mathbf{1}_n$  (resp.,  $\mathbf{0}_{mn}$  and  $\mathbf{1}_{mn}$ ) will stand for length- $n$  column vectors (resp.  $m \times n$  matrices) consisting entirely of 0's and 1's. For two matrices  $A$  and  $B$ , the matrix  $A \otimes B$  is the Kronecker (or tensor) product of  $A$  and  $B$ . For a graph  $G$  with adjacency matrix  $A$ , the characteristic polynomial of  $G$  is  $f_G(\lambda) := \det(\lambda I - A)$ . We use the standard notations  $P_n$ ,  $C_n$ ,  $S_n$ , and  $K_n$  respectively for the path, cycle, star, and complete graphs on  $n$  vertices.

## 2. THE MAIN THEOREM

Let  $G$  and  $H$  be finite simple graphs on  $m$  and  $n$  vertices, respectively, and let  $A$  and  $B$  denote their respective adjacency matrices. We begin by choosing a convenient labeling of the vertices of  $G \circ H$ . Recall that  $G \circ H$  is comprised of the  $m$  vertices of  $G$ , which we label arbitrarily using the symbols  $\{1, 2, \dots, m\}$ , and  $m$  copies  $H_1, H_2, \dots, H_m$  of  $H$ . Choose an arbitrary ordering  $h_1, h_2, \dots, h_n$  of the vertices of  $H$ , and label the vertex in  $H_i$  corresponding to  $h_k$  by the label  $i + mk$ . Below is a sample corona with the above labeling procedure:



Under this labelling the adjacency matrix of  $G \circ H$  is given by

$$A \circ B := \begin{bmatrix} A & \mathbf{1}_n^T \otimes I_m \\ \mathbf{1}_n \otimes I_m & B \otimes I_m \end{bmatrix}.$$

The goal now is to compute the eigenvalues of this corona matrix in terms of the spectra of  $A$  and  $B$ . We introduce one new invariant for this purpose.

**Definition 1.** Let  $H$  be a graph on  $n$  vertices, with adjacency matrix  $B$ . Note that, viewed as a matrix over the field of rational functions  $\mathbb{C}(\lambda)$ , the characteristic matrix  $\lambda I - B$  has determinant  $\det(\lambda I - B) = f_H(\lambda) \neq 0$ , so is invertible. The *coronal*  $\chi_H(\lambda) \in \mathbb{C}(\lambda)$  of  $H$  is defined to be the sum of the entries of the matrix  $(\lambda I - B)^{-1}$ . Note this can be calculated as

$$\chi_H(\lambda) = \mathbf{1}_n^T (\lambda I_n - B)^{-1} \mathbf{1}_n.$$

Our main theorem is that, beyond the spectra of  $G$  and  $H$ , only the coronal of  $H$  is needed to compute the spectrum of  $G \circ H$ .

**Theorem 2.** Let  $G$  and  $H$  be graphs with  $m$  and  $n$  vertices. Let  $\chi_H(\lambda)$  be the coronal of  $H$ . Then the characteristic polynomial of  $G \circ H$  is

$$f_{G \circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda)).$$

In particular, the spectrum of  $G \circ H$  is completely determined by the characteristic polynomials  $f_G$  and  $f_H$ , and the coronal  $\chi_H$  of  $H$ .

*Proof.* Let  $A$  and  $B$  denote the respective adjacency matrices of  $G$  and  $H$ . We compute the characteristic polynomial of the matrix  $A \circ B$ . For this, we recall two elementary results from linear algebra on the multiplication of Kronecker products and determinants of block matrices:

- In cases where each multiplication makes sense, we have

$$M_1 M_2 \otimes M_3 M_4 = (M_1 \otimes M_3)(M_2 \otimes M_4).$$

- If  $M_4$  is invertible, then

$$\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4) \det(M_1 - M_2 M_4^{-1} M_3).$$

Combining these two facts, we have (as an equality of rational functions)

$$\begin{aligned} f_{G \circ H}(\lambda) &= \det(\lambda I_{m(n+1)} - A \circ B) \\ &= \det \begin{pmatrix} \lambda I_m - A & -\mathbf{1}_n^T \otimes I_m \\ -\mathbf{1}_n \otimes I_m & \lambda I_{mn} - B \otimes I_m \end{pmatrix} \\ &= \det \begin{pmatrix} \lambda I_m - A & -\mathbf{1}_n^T \otimes I_m \\ -\mathbf{1}_n \otimes I_m & (\lambda I_n - B) \otimes I_m \end{pmatrix} \\ &= \det((\lambda I_n - B) \otimes I_m) \det \left[ (\lambda I_m - A) - (\mathbf{1}_n^T \otimes I_m)((\lambda I_n - B) \otimes I_m)^{-1}(\mathbf{1}_n \otimes I_m) \right] \\ &= \det(\lambda I_n - B)^m \det(\lambda I_m - A - (\mathbf{1}_n^T (\lambda I_n - B)^{-1} \mathbf{1}_n) \otimes I_m) \\ &= \det(\lambda I_n - B)^m \det((\lambda - \chi_H(\lambda))I_m - A) \\ &= f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda)). \end{aligned}$$

□

**Remark 3.** A natural question is whether or not the spectrum of  $G \circ H$  is determined by the spectra of  $G$  and  $H$ , i.e., whether knowledge of the coronal is strictly necessary. Indeed it is: Computing the coronals of the cospectral graphs  $S_5$  and  $C_4 \cup K_1$ , we have

$$\chi_{S_5}(\lambda) = \frac{5\lambda + 8}{\lambda^2 - 4} \quad \text{and} \quad \chi_{C_4 \cup K_1}(\lambda) = \frac{5\lambda - 2}{\lambda^2 - 2\lambda}.$$

Thus cospectral graphs need not have the same coronal, and hence for a given graph  $G$ , the spectra of  $G \circ S_5$  and  $G \circ (C_4 \cup K_1)$  are almost always distinct. Note that this stands in stark contrast to the situation for the Cartesian and tensor products of graphs. In both of these cases, the spectrum of the product is determined by the spectra of the components.

The unexpected simplicity of the examples in the above remark are representative of a fairly common phenomena: since the coronal  $\chi_H(\lambda) = \frac{\tilde{\chi}_H(\lambda)}{f_H(\lambda)}$  can be computed as the quotient of the sum  $\tilde{\chi}_H(\lambda)$  of the cofactors of  $\lambda I - B$  by the characteristic polynomial  $f_H(\lambda)$ , it is *a priori* the quotient of a degree  $n-1$  polynomial by a degree  $n$  polynomial. In practice, however, as in the examples in the remark, these two polynomials typically have roots in common, providing for a reduced expression for the coronal. Let us suppose that  $g(\lambda) := \gcd(\tilde{\chi}_H(\lambda), f_H(\lambda))$  has degree  $n-d$  (the gcd being taken in  $\mathbb{C}[\lambda]$ ), so that  $\chi_H(\lambda)$  in its reduced form is a quotient of a degree  $d-1$  polynomial by a degree  $d$  polynomial. Since the denominator of this reduced fraction is a factor of  $f_H(\lambda)$ , and since  $f_G$  is of degree  $m$ , each pole of  $\chi_H(\lambda)$  is simultaneously a multiplicity- $m$  pole of  $f_G(\lambda - \chi_H(\lambda))$  and a multiplicity- $m$  root of

$f_H(\lambda)^m$ . Since these contributions cancel in the overall determination of the roots of  $f_{G \circ H}(\lambda)$  in the expression

$$f_{G \circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda))$$

from Theorem 2, we can now more explicitly describe the spectrum of the corona. Namely, let  $d$  be the degree of the denominator of  $\chi_H(\lambda)$  as a reduced fraction. Then the spectrum of  $G \circ H$  consists of:

- Some “old” eigenvalues, i.e., the roots of  $f_H(\lambda)$  which are not poles of  $\chi_H(\lambda)$  (or equivalently, the roots of  $g(\lambda)$ ), each with multiplicity  $|G|$ ; and
- Some “new” eigenvalues, i.e., the values  $\lambda$  such that  $\lambda - \chi_H(\lambda)$  is an eigenvalue  $\mu$  of  $G$  (with the multiplicity of  $\lambda$  equal to the multiplicity of  $\mu$  as an eigenvalue of  $G$ .)

Since for a given  $\mu$ , solving  $\lambda - \chi_H(\lambda) = \mu$  by clearing denominators amounts to finding the roots of a degree  $d + 1$  polynomial in  $\lambda$ , the above two bullets combine to respectively provide all  $(n - d)m + m(d + 1) = m(n + 1)$  eigenvalues of  $G \circ H$ .

The following table, computed using SAGE ([11]), gives the number of graphs on  $n$  vertices whose coronal has a denominator of degree  $d$  (as a reduced fraction), as well as the average degree of this denominator, for  $1 \leq n \leq 7$ .

Table 1: Number of graphs on  $n$  vertices whose coronal has denominator of degree  $d$

$d \setminus n$	1	2	3	4	5	6	7
1	1	2	2	4	3	8	6
2		0	2	5	12	28	44
3			0	2	13	50	138
4				0	6	40	304
5					0	22	246
6						8	214
7							92
Total	1	2	4	11	34	156	1044
Average $d$	1	1	1.5	1.82	2.65	3.41	4.68
(Average $d$ )/ $n$	1	0.5	0.5	0.45	0.53	0.57	0.66

Since determining the characteristic polynomial of  $G \circ H$  from the spectra of  $G$  and  $H$  requires only the extra knowledge of the coronal of  $H$ , it remains to develop techniques for computing these coronals. In Section 3, we will develop shortcuts for these computations, but we briefly conclude this section with some more computationally-oriented approaches. A first such option is to have a software package with linear algebra capabilities directly compute the inverse of  $\lambda I - B$  and sum its entries, as done in the computations for Table 1. This seems to be computationally feasible only for rather small graphs (e.g.,  $n \leq 12$ ). A second, more graph-theoretic, option relies on a combinatorial result of Schwenk [9] to compute each cofactor of  $\lambda I - B$  individually, before summing them to compute the coronal:

**Theorem 4** (Schwenk, [9]). *For vertices  $i$  and  $j$  of a graph  $H$  with adjacency matrix  $B$ , let  $\mathcal{P}_{i,j}$  denote the set of paths from  $i$  to  $j$ . Then*

$$\text{adj}(\lambda I - B)_{i,j} = \sum_{P \in \mathcal{P}_{i,j}} f_{H-P}(\lambda).$$

Again, this approach becomes computationally infeasible fairly quickly without a method for pruning the number of cofactors to calculate. We explore this idea in the next section. Regardless, from Theorem 4, we obtain:

**Corollary 5.** *The spectrum of the corona  $G \circ H$  is determined by the spectrum of  $G$  and the spectra of the proper subgraphs of  $H$  (or more economically, only from the spectra of those subgraphs obtained by deleting paths from  $H$ ).*

### 3. COMPUTING CORONALS

In this section, we will compute the coronals  $\chi_H(\lambda)$  for several families of graphs, and hence for such  $H$  obtain the full spectrum of the corona  $G \circ H$  for any  $G$ . The principal technique exploits the regularity or near-regularity of a graph in order to greatly reduce the number of cofactor calculations (relative to those required by Theorem 4) needed to compute the coronal. In particular, we use these ideas to compute the coronals of regular graphs, complete bipartite graphs, and paths.

For graphs that are “nearly regular” in the sense that their degree sequences are almost constant, we can take advantage of linear-algebraic symmetries to compute the coronals. We begin with two concrete computations, those corresponding to regular and complete bipartite graphs, before extracting the underlying heuristic and applying it to the coronal of path graphs. The case of regular graphs, first addressed in [1], is particularly straight-forward from this viewpoint.

**Proposition 6** (Regular Graphs). *Let  $H$  be  $r$ -regular on  $n$  vertices. Then*

$$\chi_H(\lambda) = \frac{n}{\lambda - r}.$$

*Thus for any graph  $G$ , the spectrum of  $G \circ H$  is precisely:*

- *Every non-maximal eigenvalue of  $H$ , each with multiplicity  $|G|$ .*
- *The two eigenvalues*

$$\frac{\mu + r \pm \sqrt{(r - \mu)^2 + 4n}}{2}$$

*for each eigenvalue  $\mu$  of  $G$ .*

*Proof.* Let  $B$  be the adjacency matrix of  $H$ . By regularity, we have  $B\mathbf{1}_n = r\mathbf{1}_n$ , and hence  $(\lambda I - B)\mathbf{1}_n = (\lambda - r)\mathbf{1}_n$ . Cross-dividing and multiplying by  $\mathbf{1}_n^T$ ,

$$\chi_H(\lambda) = \mathbf{1}_n^T (\lambda I - B)^{-1} \mathbf{1}_n = \frac{\mathbf{1}_n^T \mathbf{1}_n}{\lambda - r} = \frac{n}{\lambda - r}.$$

The only pole of  $\chi_H(\lambda)$  is the maximal eigenvalue  $\lambda = r$  of  $H$ , and the “new” eigenvalues are obtained by solving  $\lambda - \frac{n}{\lambda - r} = \mu$  for each eigenvalue  $\mu$  of  $G$ .  $\square$

It is noteworthy that all  $r$ -regular graphs on  $n$  vertices have the same coronal, especially given that the cofactors of the corresponding matrices  $(B - \lambda I)^{-1}$  appear to be markedly dissimilar. The simplicity of this scenario, and the easily checked observation that cospectral regular graphs must have the same regularity, lead to the following corollary. We will make use of this corollary in the final section.

**Corollary 7.** *Cospectral regular graphs have the same coronal.*

As a second class of examples, we compute the coronals of complete bipartite and  $n$ -partite graphs.

**Proposition 8** (Complete Bipartite Graphs). *Let  $H = K_{p,q}$  be a complete bipartite graph on  $p + q = n$  vertices. Then*

$$\chi_H(\lambda) = \frac{n\lambda + 2pq}{\lambda^2 - pq}.$$

For any graph  $G$ , the spectrum of  $G \circ H$  is given by:

- The eigenvalue 0 with multiplicity  $m(n - 2)$ ; and
- For each eigenvalue  $\mu$  of  $G$ , the roots of the polynomial

$$x^3 - \mu x^2 - (p + q + pq)x + pq(\mu - 2).$$

*Proof.* Let  $B = \begin{bmatrix} \mathbf{0}_{p,p} & \mathbf{1}_{p,q} \\ \mathbf{1}_{q,p} & \mathbf{0}_{q,q} \end{bmatrix}$  be the adjacency matrix of  $K_{p,q}$  and let  $X = \text{diag}((q + \lambda)I_p, (p + \lambda)I_q)$  be the diagonal matrix with the first  $p$  diagonal entries being  $(q + \lambda)$  and the last  $q$  entries being  $(p + \lambda)$ . Then  $(\lambda I - B)X\mathbf{1}_n = (\lambda^2 - pq)\mathbf{1}_n$ , and so

$$\chi_H(\lambda) = \mathbf{1}_n^T (\lambda I - B)^{-1} \mathbf{1}_n = \frac{\mathbf{1}_n^T X \mathbf{1}_n}{\lambda^2 - pq} = \frac{(p + q)\lambda + 2pq}{\lambda^2 - pq}.$$

Thus the coronal has poles at both of the non-zero eigenvalues  $\pm\sqrt{pq}$  of  $K_{p,q}$ , leaving only the eigenvalue 0 with multiplicity  $p + q - 2$ . Finally, solving  $\lambda - \chi_H(\lambda) = \mu$  gives the new eigenvalues in the spectrum as stated in the proposition.  $\square$

**Remark 9.** It might be tempting in light of Propositions 6 and 8 to hope that the degree sequence of a graph determines its coronal. This too, like the analogous conjecture stemming from cospectrality (Remark 3), turns out to be false: The graphs  $P_5$  and  $K_2 \cup K_3$  have the same degree sequence, but we find by direct computation that

$$\chi_{P_5}(\lambda) = \frac{5\lambda^2 + 8\lambda - 1}{\lambda^3 - 3\lambda} \quad \chi_{K_2 \cup K_3}(\lambda) = \frac{5\lambda - 7}{\lambda^2 - 3\lambda + 2}.$$

Similar to the complete bipartite computation, we have the following somewhat technical generalization to complete  $k$ -bipartite graphs.

**Proposition 10.** *Let  $H$  be the complete  $k$ -partite graph  $K_{n_1, n_2, \dots, n_k}$ . Then*

$$\chi_H(\lambda) = \left( \frac{\prod_{j=1}^k (n_j + \lambda)}{\sum_{j=1}^k n_j \prod_{\substack{i=1 \\ i \neq j}}^k (n_i + \lambda)} - 1 \right)^{-1} = \frac{\sum_{j=1}^k j C_j \lambda^{k-j}}{\lambda^k - \sum_{j=2}^k (j-1) C_j \lambda^{k-j}}$$

where  $C_j$  is the sum of the  $\binom{k}{j}$  products of the form  $n_{i_1} n_{i_2} \cdots n_{i_j}$  with distinct indices.

*Proof.* Let  $B$  be the adjacency matrix of  $H$ , let  $g_j(\lambda) = \prod_{\substack{i=1 \\ i \neq j}}^k (n_i + \lambda)$ , and let  $X$  be the block diagonal matrix whose  $j$ -th block is  $g_j I_{n_j}$ . Then

$$(\lambda I_n - B)X \mathbf{1}_n = \left[ \prod_{i=1}^k (n_i + \lambda) - \sum_{j=1}^k n_j g_j \right] \mathbf{1}_n.$$

Solving as in the bipartite case, we find

$$\begin{aligned} \chi_H(\lambda) &= \mathbf{1}_n^T (\lambda I - B)^{-1} \mathbf{1}_n = \frac{\mathbf{1}_n^T X \mathbf{1}_n}{\prod_{i=1}^k (n_i + \lambda) - \sum_{j=1}^k n_j g_j} \\ &= \frac{\sum_{j=1}^k n_j g_j}{\prod_{i=1}^k (n_i + \lambda) - \sum_{j=1}^k n_j g_j}, \end{aligned}$$

which gives the result.  $\square$

The proof technique for the last two propositions generalizes to “nearly regular” graphs, by which we mean graphs  $H$  for which all but a small number of vertices have the same degree  $r$ . In this case, we can write

$$(\lambda I - B) \mathbf{1}_n = (\lambda - r) \mathbf{1}_n + \mathbf{v},$$

where  $\mathbf{v} = (v_i)$  is a vector consisting mostly of 0’s. This gives

$$(\lambda I - B)^{-1} \mathbf{1}_n = \frac{1}{\lambda - r} [\mathbf{1}_n - (\lambda I - B)^{-1} \mathbf{v}],$$

and thus, using the adjugate formula for the determinant,

$$\chi_H(\lambda) = \mathbf{1}_n^T (\lambda I - B)^{-1} \mathbf{1}_n = \frac{1}{\lambda - r} \left[ n - \frac{1}{f_H(\lambda)} \sum_{1 \leq i, j \leq n} v_i C_{i,j} \right],$$

where  $C_{i,j}$  denotes the  $(i, j)$ -cofactor of  $\lambda I - B$ . Since  $v_i$  is zero for most values of  $i$ , we have an effective technique for computing coronals if we can compute a small number of cofactors (as opposed to, in particular, computing *all* of the cofactors and using Theorem 4). For example, if we let  $f_n = f_{P_n}(\lambda)$  be the characteristic polynomial of the path graph  $P_n$  on  $n$  vertices (by convention, set  $f_0 = 1$ ), we can compute coronals of paths as follows:

**Proposition 11** (Path Graphs). *Let  $H = P_n$ . Then*

$$\chi_H = \frac{n f_n - 2 \sum_{j=0}^{n-1} f_j}{(\lambda - 2) f_n}.$$

*Proof.* In the notation of the discussion preceding the proposition, we take  $r = 2$  and  $\mathbf{v} = [1 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 1]^T$ . Further, we note that an easy induction argument using cofactor expansion gives  $C_{j,1} = C_{j,n} = f_{j-1}$ . Thus we obtain

$$\chi_{P_n}(\lambda) = \frac{1}{\lambda - 2} \left[ n - \frac{1}{f_n} \sum_{j=1}^n (C_{j,1} + C_{j,n}) \right] = \frac{1}{\lambda - 2} \left[ n - \frac{2}{f_n} \sum_{j=0}^{n-1} f_j \right],$$

from which the result follows.  $\square$

From this, we easily calculate the coronals for the first few path graphs:

Table 2: The coronals  $\chi_{P_n}(\lambda)$  for  $1 \leq n \leq 7$ 

$n$	1	2	3	4	5	6	7
$\chi_{P_n}(\lambda)$	$\frac{1}{\lambda}$	$\frac{2}{\lambda-1}$	$\frac{3\lambda+4}{\lambda^2-2}$	$\frac{4\lambda+2}{\lambda^2-\lambda-1}$	$\frac{5\lambda^2+8\lambda-1}{\lambda^3-3\lambda}$	$\frac{6\lambda^2+4\lambda-4}{\lambda^3-\lambda^2-2\lambda+1}$	$\frac{7\lambda^3+12\lambda^2-6\lambda-8}{\lambda^4-4\lambda^2+2}$

**Remark 12.** This particular example can also be computed using Theorem 4: For  $i$  and  $j$  distinct, there is a unique path  $[i, j]$  from vertex  $i$  to vertex  $j$ , so the sum in the theorem reduces to a single term:

$$\text{adj}(\lambda I - B)_{ij} = f_{P_n - [i, j]} = f_{i-1}f_{n-j}.$$

Similarly, we find  $\text{adj}(\lambda I - B)_{ii} = f_{i-1}f_{n-i}$ . Summing over all the cofactors gives

$$\chi_{P_n}(\lambda) = \frac{1}{f_n} \left( \sum_{i=1}^n f_{i-1}f_{n-i} + 2 \sum_{i,j=1}^n f_{i-1}f_{n-j} \right),$$

which reduces to the result computed in Proposition 11 after some arithmetic.

#### 4. COSPECTRALITY

At the end of [1], the authors note that if  $G_1$  and  $G_2$  are cospectral graphs, then  $G_1 \circ K_1$  and  $G_2 \circ K_1$  are also cospectral, and that (by repeated coronation with  $K_1$ ) this leads to an infinite collection of cospectral pairs. Armed with the characteristic polynomial

$$f_{G \circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda))$$

of the corona (Theorem 2), we can greatly generalize this observation on two fronts.

**Corollary 13.** *If  $G_1$  and  $G_2$  are cospectral, and  $H$  is any graph, then  $G_1 \circ H$  and  $G_2 \circ H$  are cospectral. Further, if  $H_1$  and  $H_2$  are cospectral and  $\chi_{H_1} = \chi_{H_2}$ , and  $G$  is any graph, then  $G \circ H_1$  and  $G \circ H_2$  are cospectral.*

We remark that examples of this second type do indeed exist. Define the *switching graph*  $\text{Sw}(T)$  of a tree  $T$  with adjacency matrix  $A_T$  to be the graph with adjacency matrix

$$A_{\text{Sw}(T)} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A_T + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A_{\overline{T}},$$

and let  $T_1$  and  $T_2$  be non-isomorphic cospectral trees with cospectral complements (note that by [5], generalizing [8], “almost all” trees admit a cospectral pair with cospectral complement). Then the switching graphs  $\text{Sw}(T_1)$  and  $\text{Sw}(T_2)$  are non-isomorphic cospectral regular graphs (see [6], Construction 3.7), and also have the same coronal by Corollary 7. Corollary 13 now implies that for *any* graphs  $G$  and  $H$ , we have the cospectral pair  $G \circ \text{Sw}(T_1)$  and  $G \circ \text{Sw}(T_2)$  and the cospectral pair  $\text{Sw}(T_1) \circ H$  and  $\text{Sw}(T_2) \circ H$ . This gives, for example, infinitely many cospectral pairs of graphs with any given graph  $G$  as an induced subgraph.

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